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Gravitational Waves in an Effective Field Theory of Gravity and Electromagnetism

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Abstract

We investigate the propagation of metric and electromagnetic fluctuations on a perturbed Reissner-Nördstorm background in an low energy effective field theory of gravity and electromagnetism. We derive a modified Zerilli equation for metric perturbations for this theory.

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Notation and Conventions

The following conventions shall be used throughout this work unless explicitly otherwise stated:

Spacetime is 4-dimensional with a metric signature $\{-, +, +, +\}$.

The units used are $\hbar = c = k_e = 1$

The reduced Planck mass is defined as $M_{pl}^2 = \frac{1}{8\pi G}$

We use the Einstein summation convention for repeated indices

We indicate partial differentiation by $\frac{\partial}{\partial\mu}$ or ∂_μ and covariant differentiation by ∇_μ

Chapter 1

Introduction

In classical General Relativity, massless particles travel on null geodesics. It is said they are travelling at the speed of light. This often taught in a tautological way- each defining the other. We shall try to be precise and define the speed of a via the lightcone of the effective metric that our fields propagate. In General Relativity, the effective metric of massless particles coincide with the actual metric[1]. This is essentially a result of the strong equivalence principle. The strong equivalence principle can split into two statements [16]. The statement that at each point in spacetime there exists a local Minkowski frame (Local Inertial Frame), known as the weak equivalence principle. We like to think of spacetime as a Lorentzian manifold so this is essentially a differential geometry statement. The second statement is that the laws of physics are equivalent in Local Inertial Frames established at different points in spacetime. The effect of this is to exclude all explicit curvature couplings. i.e all matter terms are minimally coupled and higher derivative curvature terms are omitted. General Relativity is such a theory and explicitly it can be shown that the two derivative nature of General Relativity and the weak equivalence principle cause luminality [12, 14]. In cases where the the strong equivalence principle does not hold the physical light cone can differ from the geometric null cone. We are particularly interested how this might occur from a quantum theory perspective. It is well known that in QED the speed of the photon may be modified in a background gravitational field [10, 17–26]. This remarkable effect deserves some explanation. During propagation, the photon can produce short-lived virtual electron-positron

[1] By this, we mean the metric providing the geometry

pairs. The result of these is to give a 'size' to the photon on the order of the Compton wavelength of the electron. The photon is now sensitive to curvature of the background gravitational field. It is these tidal effects which lead to a modification of the photon speed.

To investigate these sort of effects we shall work in the framework of low energy effective theories. We would like to briefly motivate the use of this framework. You may have heard what has become a popular phrase that general relativity and quantum mechanics are not compatible. Specifically we cannot treat general relativity as a quantum theory because it is non-renormalizable as a field theory. This is not necessarily true. In the low energy limit, non-renormalizable theories can be predictive if we can use some perturbative expansion to only consider a finite amount of terms in the Lagrangian. Therefore in this framework it is consistent to consider a quantum theory of gravity. Of course this is no longer acceptable at high energies such as at the singularity of a black hole, but this problem is due to a lack of a theory of everything. There is a huge amount of work devoted to finding such a theory, the most famous example being String Theory. One of the beauties of effective field theory is that we can stay agnostic about the high energy theory.

In the case a of low energy QED, it is the explicit curvature-Maxwell couplings which can lead to a subluminal or superluminal photon propagation. This has also more recently been done for the graviton in the low energy EFT of gravity for different backgrounds [11,12]. I should stress that it is not believed that the superluminal velocities violate causality [13]. Additionally [14] discusses why spacelike propagation might be allowed in backgrounds that spontaneously break Lorentz invariance even though the theory itself is Lorentz invariant.

In the framework of low energy effective field theories, I will investigate the speed of gravitational waves propagating on a Reissner-Nördstrom like background. This has been done for a the Schwarzchild-like background [11] and I shall extend to Reissner-Nördstrom-like backgrounds. The main difference is the inclusion of matter in the form of the electromagnetic field. The low energy effective field theory will contain a host of higher derivative curvature terms, curvature-matter couplings, etc. which could potentially lead to changes in the speed of both photons and gravitons. We will consider the low energy effective action of gravity and electromagnetism including operators which have up to four derivatives in the light fields $g_{\mu\nu}$ and A_μ. We find that the leading order corrections to the action coming from these four derivative operators can be reduced to only three terms via a perturbative redefinition of the metric. Additionally, we decide to omit operators which break the minimal coupling of photons to keep the speed of photons luminal (as compared to GR). With this effective action, we do not find a departure of the radial speed of gravitational waves from unity as compared to the speed of the photon.

The work is organized as follows. In chapter 2, I shall give a brief introduction to low energy effective field theories and construct the low energy effective field theory of gravity and electromagnetism. In chapter 3, I shall study the slight deviations that the low energy effective field theory has on the classical Reissner-Nordstrom solution. In chapter 4, I shall derive a modified Zerilli equation for the propagation of metric perturbations on top of the perturbed EFT-Reissner-Nordstrom background solution.

Chapter 2

Effective Field Theory of Gravity and Electromagnetism

2.1 The Low Energy Effective Action Basics

In this section, I will formally define the low energy effective field theory. Much of this section follows chapter 2 of an *An Introduction to Effective Field Theories* by Cliff Burgess [7].

We start by considering a field theory for N real scalar fields with a classical action $S(\phi)$. We then can define the generating functional $Z[J]$ for these fields where $J_a(x)$ are currents sourcing the fields:

$$Z[J] = \int D\phi \, e^{iS(\phi)+i\int d^4x J_a(x)\phi^a(x)}. \tag{2.1}$$

Using the Gell-Mann and Low theorem, we can calculate the vacuum expectation values of all correlation functions from the quantity.

$$\frac{1}{Z[0]}\frac{(-i)^n\delta^n}{\delta J_{a_1}(x_1)\ldots\delta J_{a_n}(x_n)}Z[J] = \langle\Omega|\hat{T}\hat{\phi}_{a_1}(x_1)\ldots\hat{\phi}_{a_n}(x_n)|\Omega\rangle. \tag{2.2}$$

Correlation functions are required to compute the S-matrix or scattering amplitudes and therefore, from a practical point of view, determining the generating functional gives us the essential information of a theory. It is also convenient to define another generating functional $W[J]$.

$$\frac{Z[J]}{Z[0]} = e^{iW[J]}. \tag{2.3}$$

The simplest way to distinguish between these two quantities is in perturbation theory. We can expand the generating functionals as a series of Feynman diagrams. In this language $\frac{Z[J]}{Z[0]}$ can be described by all possible Feynman excluding vacuum bubbles. $W[J]$ can be described by all connected Feynman diagrams also excluding vacuum bubbles. However there is still another simpler quantity that describes our system. First we define a quantity $\bar{\phi}^a$ which we can think as the vacuum expectation value of the field in the presence of the current.

$$\bar{\phi}^a = \langle\Omega|\hat{T}\hat{\phi}^a(x)|\Omega\rangle_c = \frac{\delta W[J]}{\delta J_a}. \tag{2.4}$$

We define the 1 particle irreducible effective action (or quantum effective action) by performing a Legendre transform. In the same way as $Z[J]$ and $W[J]$ we can think of this quantity being determined by a class of Feynman diagrams. In this case is it generated by connected diagrams that cannot be separated into two valid diagrams by splitting an internal line. (Also excluding vacuum bubbles).

$$\Gamma[\bar{\phi}] = W[J] - \int d^4x \; J_a(x)\bar{\phi}^a(x). \tag{2.5}$$

At this point it is easy to do some algebraic gymnastics in order to give the path integral definition of $\Gamma[\bar{\phi}]$, reinstating factors of $\hbar$ into the answer.

$$e^{\frac{i\Gamma[\bar{\phi}]}{\hbar}} = N \int D\phi \; e^{\frac{i}{\hbar}[S(\phi)+\int d^4x \; J_a(x)(\phi^a - \bar{\phi}^a)]}. \tag{2.6}$$

With these quantities defined, it is possible to adjust this formalism for the low energy effective theory case. For simplicity take a theory which contains two different mass/energy scales: $m_l \ll m_h$. We can then split the quantum field into a low energy and high energy part by choosing a cut-off Λ such that $m_l \ll \Lambda \ll m_l$. We can divide the low and high energy fields by projecting on the states with $E < \Lambda$ using the projection operator denoted by P_Λ

$$l^a = P_\Lambda \phi^a P_\Lambda, \tag{2.7}$$

$$h^a = (1 - P_\Lambda)\phi^a(1 - P_\Lambda). \tag{2.8}$$

We now may define the low-energy generating functional $Z_{le}[J]$. One may be tempted to define this by just replacing ϕ with l, however this would not be correct because it would ignore the effects of the high energy fields. What we want to do is to find a generating functional that will reproduce the correlation functions we would calculate in low energy limit of the full theory. The way to do this is to restrict the currents in the generating functional to be slowly varying (as set by the cutoff). Since these slowly varying currents only couple to the low energy fields, we have found a theory which only produces correlation functions of the low energy fields yet is still able to capture the influence of the high energy fields.

$$Z_{le}[j] = Z[j, J = 0], \tag{2.9}$$

$$Z_{le}[j] = \int D\phi \, e^{iS(\phi)+i\int d^4xj(x)\phi(x)} = \int D\phi \, e^{iS(\phi)+i\int d^4xj(x)l(x)}, \tag{2.10}$$

$$= \int DhDl \, e^{iS(l+h)+i\int d^4xj(x)l(x)}. \tag{2.11}$$

We then can follow the same steps as before in order to define the connected low energy generating functional $W_{le}[j]$ and the 1 light particle irreducible action $\Gamma_{le}[\bar{l}]$. We first define a quantity $\bar{l}$ which we can think of as the vacuum expectation value of the light field:

$$\bar{l} = \frac{\delta W_{le}[j]}{\delta j}. \tag{2.12}$$

Then the low energy forms of eq(3) and eq(5)

$$\frac{Z_{le}[j]}{Z_{le}[0]} = e^{iW_{le}[j]}, \tag{2.13}$$

$$\Gamma_{le}[\bar{l}] = W_{le}[j] - \int d^4x \, j(x)\bar{l}^a(x). \tag{2.14}$$

Finally, we can give the path integral definition for $\Gamma_{le}[\bar{l}]$, again reinstating the factors of $\hbar$.

$$e^{\frac{i}{\hbar}\Gamma_{le}[\bar{l}]} = N\int DlDh\, e^{\frac{i}{\hbar}(S(l+h)+\int d^4xj(x)(l(x)-\bar{l}(x))} \tag{2.15}$$

$$= N\int Dl\, e^{\frac{i}{\hbar}(S_w(l)+\int d^4xj(x)(l(x)-\bar{l}(x))}, \tag{2.16}$$

where:

$$e^{\frac{i}{\hbar}S_w(l)} = \int Dh\, e^{\frac{i}{\hbar}S(l+h)}. \tag{2.17}$$

The quantity $S_w(l)$ is what we call the Wilson action or low energy effective action. This is the main result of this section which is essential for the rest of this paper, therefore is worth a few additional comments.

Firstly, this quantity captures the full low energy influence of the heavy fields on the light fields: the heavy fields appear nowhere else. The functional integration of the heavy fields, resulting in a quantity only dependant on the light degrees of freedom, is known as integrating out the heavy degrees of freedom. A consequence of this that I would like to highlight is that the Wilson action defined in this way is not an approximation: it is an alternative description of the dynamics of the low energy fields which is valid only up to a certain energy scale.[1]

Secondly, we see that the Wilson action appears in the 1LPI action in exactly the same way in which the classical action appears in the 1PI irreducible action of the 'full' theory. This suggests that our full theory may actually appear as a low energy effective theory of an even higher energy theory [7]. Indeed it is now thought that the Standard Model and General Relativity are just low energy approximations for some high energy theory such as String theory.

A fair question which could be asked at this point is whether this is practically useful. Even if we know some low energy effective action exists, how do we find it and how do we know that we can perform useful calculations with it? In general the process of integrating out will result in a non-local and non-renormalizable action [15]. However this is not actually a huge problem. We can replace the non-local interactions by a series of local interactions which give the same physics at low energy. The essential point is in doing this we have modified the high-energy behaviour

[1]However, for practical applications we always take a approximation of this quantity

of the theory, and hence it is non-sensical to use this theory as a description of high energy behaviour. An upper bound on the breakdown of the theory is given by the cutoff Λ.

Returning to the problem of replacing the non-local interaction with a series of local interactions. This can be done formally using the operator product expansion [15]. Since we are not doing this explicitly, a more intuitive picture given by [30] suffices. Consider a scattering process between two light particles at low energy. We can produce a virtual heavy particle, however by the uncertainty principle its range is only $\frac{1}{m}$, where m is the mass of the heavy particle. If we are working a low energy (i.e much less than M), we cannot probe distances smaller than this and hence interactions will look local. Another way to look at this is to consider this process if we did the full theory calculation and then took the low energy limit. For simplicity, suppose our virtual particle is a heavy boson with a propagator $\propto \frac{1}{p^2-m^2}$. We then do a Taylor expansion of this propagator:

$$\frac{1}{p^2-m^2} = -\frac{1}{m^2} - \frac{p^2}{m^4} - \frac{p^4}{m^6} + \dots \tag{2.18}$$

In position space, we can identify the $\frac{1}{m^2}$ terms with a delta function and the p^i terms as being generated by derivatives (of the low energy fields). Hence we expect to be able to replace its effects in the Lagrangian as a series of local terms. This example is actually illustrative in a few other ways. Firstly, it indicates that each derivative will appear with a $1/m$ in a expansion of our low energy Lagrangian, where m is the mass of the heavy particle that is integrated out. Secondly, suppose we have a hierarchy of particles that have been integrated out.Each derivative will appear with a $1/m_i$ where i indexes the particle type. Therefore, the biggest contribution will always be from the lightest particle that has been integrated out.

With this intuitive picture in mind, how can we construct our low energy effective Lagrangian? The essential idea is to write down the most general possible Lagrangian consistent with symmetries and then perform dimensional analysis on it. I will briefly explain how to do this for a scalar field in flat space, before presenting a much more thorough version for the effective field theory of gravity and photons. Suppose our low energy theory field content is a light scalar field. We know that we can write our action as a power series over some heavy mass scale. We write our action as follows:

$$S = f^4 \int d^4x c_n O_n(\frac{\phi}{v}, \frac{\partial\phi}{Mv}). \tag{2.19}$$

Where we suppose that the scales f, v, M are much greater than the energy of our light particles and this makes our dimensionless couplings $c_i \sim O(1)$[2]. To stay general, these do not have to depend on each other, however we expect them to be related the mass scales integrated out in the problem. As a nod to the above paragraphs, we have labelled the scale controlling the derivative by M, which we would expect to be the lightest mass scale integrated out. We therefore have a Lagrangian whose higher dimensional operators are suppressed. Additionally, we do not have to worry about renormalizability because we have a manifestly finite theory, since all loop integrals have a momentum cutoff. However, one must be careful with these arguments. Consider Feynman diagrams with loops. We have high energy virtual particles (up to the cutoff) running through these loops. When we evaluate the diagram, the loop integrals may cause the answer to be multiplied by factors Λ. Indeed, we may have the situation in which more complicated diagrams with more loops enhance the answer. Although each diagram is finite, we cannot actually calculate anything with this Lagrangian because we have an infinite amount of diagrams to deal with. For some theories, the solution to this problem is to prove that more complicated diagrams will also be suppressed. This technique was first used by Weinberg in the context of chiral perturbation theory [3]. In the next section we will construct the Lagrangian for gravity and photons and then prove its applicability in this way.

2.2 The Effective Action for Gravity and Electromagnetism

In this section, we consider the the low energy effective field theory of gravity and electromagnetism. The low energy degrees of freedom are the metric $g_{\mu\nu}$ and the Maxwell field A_μ. We require the action to be both $U(1)$ gauge invariant and diffeo-

[2]In the next section, we will show that we can relax this assumption for the field energy.

morphism invariant. This is a statement that physics should not depend on gauge or coordinate system chosen. Because of these requirements, it is simpler to consider a Lagrangian made out of scalar combinations of $g_{\mu\nu}, R, R_{\mu\nu}, R_{\alpha\beta\mu\nu}, F_{\mu\nu}$ and their covariant derivatives. This will capture all the possible terms along with concisely maintaining gauge and diffeomorphism invariance. We write our Lagrangian as a derivative expansion, up to 4 derivatives acting on our fields. Before doing this explicitly I will mention types of terms which we need not include in our Lagrangian.

2.2.1 Cosmological Constant Term

This is the constant term in our Lagrangian. Theoretically this term should be included, however its calculated size is massive compared to the observed value, which is almost zero. This is known as the cosmological constant problem. For an interesting discussion on this topic see [9]. For the purpose of this analysis I shall set it to zero.

2.2.2 Terms Forbidden by Symmetries

I have already mentioned that our Lagrangian must be both diffeomorphism and gauge invariant. This means that we can ignore terms which break this such as $\Gamma^{\mu}_{\alpha\beta} A_\mu g^{\alpha\beta}$. Other terms ignored are those which are zero because of the symmetry properties of tensor. For example $F_{\mu\nu}R^{\mu\nu}$ since $F_{\mu\nu}$ and $R_{\mu\nu}$ are antisymmetric and symmetric respectively in the $(\mu\nu)$ indices.

2.2.3 Total Derivatives

Another type of operator which can be omitted are total derivatives. By Stokes's theorem, these will lead to boundary terms when integrated. In this paper, I shall work in a space-time manifold that has no boundary, thus surface terms vanish. If one were to work on a manifold with a boundary, it would not be possible to disregard these terms. To four derivatives, the only covariant terms containing a covariant derivative acting on a curvature term are:

$$\Box R. \tag{2.20}$$

Since this is a total derivatives, it can be omitted. Up to four derivatives, the only relevant terms containing a covariant derivative acting on the field strength tensor must be of the form $(\nabla F)(\nabla F)$. To see this, note that any term containing just one field strength tensor will necessarily be a total derivative. No terms containing two or three field strength tensors with just one covariant derivative will have an even number of indices. Therefore the only allowed terms must have two field strength tensors and two covariant derivatives. Suppose we had a term of the form $F(\nabla\nabla F)$. This can be written in the form $\nabla(F\nabla F) - (\nabla F)(\nabla F)$. Since total derivatives are omitted, all terms can be written in the form $(\nabla F)(\nabla F)$.

Putting in the explicit indices, the possible terms of this form are:

$$\nabla_\alpha F^{\mu\nu} \nabla^\alpha F_{\mu\nu} \tag{2.21}$$

$$\nabla_\alpha F^{\alpha\beta} \nabla_\mu F^\mu{}_\beta \tag{2.22}$$

$$\nabla_\alpha F^{\beta\gamma} \nabla_\beta F^\alpha{}_\gamma \tag{2.23}$$

However, these are not yet independant. Using the Ricci identities and the Maxwell equations, these terms may be recast into non-derivative form and the combination $\nabla_\alpha F^{\mu\nu} \nabla^\alpha F_{\mu\nu}$ [4].

2.2.4 Field Redefinition

Another way to remove redundant operators is via a perturbative redefinition of the field. A nice description of how this is generally done for a real scalar field is given in section 2.5 of [7]. For gravity, this is done via a perturbative redefinition of the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \epsilon Y_{\mu\nu}, \tag{2.24}$$

where $Y_{\mu\nu}$ is expressed in terms of combinations of the same quantities used to build the Lagrangian ($g_{\mu\nu}, R, R_{\mu\nu}, R_{\alpha\beta\mu\nu}, F_{\mu\nu}$) and ϵ is a small parameter governing the expansion in the Lagrangian. It is simpler to perform this redefinition after first removing other redundant operators, so I shall wait to perform this until explicitly defining the action in section 2.2.6.

2.2.5 Gauss-Bonnet Term

Starting with the well known equation

$$R^2_{GB} = R^2_{\mu\nu\alpha\beta} - 4R^2_{\mu\nu} + R^2, \tag{2.25}$$

where R^2_{GB} is the Gauss-Bonnet term. In four dimensions, the Gauss-Bonnet terms is topological and therefore does not contribute to the equations of motion. This allows us to omit the $R^2_{\mu\nu\alpha\beta}$ term.

2.2.6 Explicit Form of Action

The most general action of the theory up to four derivatives in the light fields can be written explicitly as

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Bigg[&\frac{M_{pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_1R^2 + c_2R_{\mu\nu}R^{\mu\nu} + \frac{c_4}{M_{pl}^2}F_{\mu\nu}F^{\mu\nu}R \\ &+ \frac{c_5}{M_{pl}^2}F^{\mu\rho}F^{\nu}{}_{\rho}R_{\mu\nu} + \frac{c_6}{M_{pl}^2}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\rho\sigma} + \frac{c_7}{M_{pl}^4}F_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ &+ \frac{c_8}{M_{pl}^4}F_{\mu\nu}F^{\nu\alpha}F_{\alpha\beta}F^{\alpha\mu} + \frac{c_9}{M_{pl}^2}\nabla_\alpha F^{\alpha\beta}\nabla_\mu F^{\mu}{}_{\beta}\Bigg]. \end{aligned} \tag{2.26}$$

Here the role of M_{pl}^2 is to make the coupling constants dimensionless, rather than setting their scale to be of similar order. I will justify why I choose to expand to the Lagrangian in the number of derivatives in section 2.3. We now perform a field redefinition of the metric to remove further redundant operators. Suppose we redefine $g_{\mu\nu} \to g_{\mu\nu} + \frac{2}{M_{pl}^2}Y_{\mu\nu}$. $Y_{\mu\nu}$ will contain at least two derivatives so up to four derivatives, it will modify the action by

$$\delta S = \frac{2}{M_{pl}^2}\int d^4x\left(\frac{M_{pl}^2}{2}(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}) - T^{\mu\nu}\right)Y_{\mu\nu} \tag{2.27}$$

$$= \int d^4x\left(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - \frac{2}{M_{pl}^2}(F^{\mu\rho}F^{\nu}{}_{\rho} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})\right)Y_{\mu\nu} \tag{2.28}$$

If we choose $Y_{\mu\nu} = (d_1R_{\mu\nu} + d_2Rg_{\mu\nu} + d_3\frac{1}{2M_{pl}^2}F_{\alpha\beta}F^{\alpha\beta}g_{\mu\nu} + d_4\frac{1}{2M_{pl}^2}F_{\mu}{}^{\rho}F_{\nu\rho})$

Then the coefficients in our initial action change as follows:

$$
\begin{aligned}
c_1 &\to c_1 - \tfrac{1}{2}d_1 - d_2 & c_6 &\to c_6 \\
c_2 &\to c_2 + d_1 & c_7 &\to c_7 + \tfrac{1}{4}d_4 \\
c_4 &\to c_4 + \tfrac{1}{2}d_1 - \tfrac{1}{2}d_3 - \tfrac{1}{4}d_4 & c_8 &\to c_8 - d_4 \\
c_5 &\to c_5 + \tfrac{1}{2}d_4 - 2d_1 & c_9 &\to c_9
\end{aligned}
$$

By specifying the d_i coefficients we can remove some of these higher-dimensional operators. It is important to note that performing a field redefinition, which is just a reparameterization of field variables, should not change any physical observables. It is a method of putting our initial action in the most convenient form, or alternatively choosing the simplest basis of higher-dimensional operators (to a certain order) for the task ahead. A corollary of this point is there is no single right basis to use. For example, we could use the whole effective action in (2.26), or any redefinition using any combination of $d_i's$ that one wants. The basis that we have chosen:

$$d_1 = 2c_5 + c_8 \tag{2.29}$$

$$d_2 = \frac{1}{8}(8c_1 - 2c_5 - c_8) \tag{2.30}$$

$$d_3 = \frac{1}{4}(8c_4 + 2c_5 - c_8) \tag{2.31}$$

$$d_4 = c_8 \tag{2.32}$$

With these choices, the effective action becomes:

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\Bigg[&\frac{M_{pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_2' R_{\mu\nu}R^{\mu\nu} + \frac{c_6'}{M_{pl}^2}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\rho\sigma} \\
&+ \frac{c_7'}{M_{pl}^4}F_{\mu\nu}F^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \frac{c_9'}{M_{pl}^2}\nabla_\alpha F^{\alpha\beta}\nabla_\mu F^{\mu}{}_{\beta}\Bigg]
\end{aligned} \tag{2.33}
$$

This is still not the effective action that I shall be working with for the rest of my analysis. In [10] it is shown that the effect of $\nabla_\alpha F^{\alpha\beta}\nabla_\mu F^{\mu}{}_{\beta}$ is actually suppressed compared to the $R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\rho\sigma}$ term and therefore can be omitted. For simplicity, I shall also set $c_6' = c_7' = 0$. Generally this will not be the case, however there are some good reasons to do this. Firstly, we would expect these terms will affect the effective metric seen by the photon. Ideally we would like the speed of photons to stay at unity as comparison to the speed of gravitational waves. Secondly, the F^4 term, although it will perturb the background metric, will not affect the propagation of gravitational waves and hence is not important for determining changes in their speed. Unfortunately this will not also be true for the RF^2 term. The third reason is that despite adding computational complexity and technical difficulty of both the photon and graviton departing from unity, using these terms does not change the approach or the analysis much.

Therefore the effective action that I shall be using is:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + cR_{\mu\nu}R^{\mu\nu}\right]. \tag{2.34}$$

2.3 Power Counting

In this section, I will give a power counting argument which is used to see how the energy scales present in the effective action will enter into observables e.g scattering amplitudes. I will use it to justify the use of the effective action (2.26), and in particular the use of a derivative rather than dimensional expansion of the effective Lagrangian. The arguments I shall give are based on those in [7, 8] adapted to this specific scenario.

The low energy degrees of freedom, as before, are the metric $g_{\mu\nu}$ and the Maxwell field A_μ. Since we are interested in observables involving gravitons it is sensible to expand the metric into a background metric and fluctuations[3].

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{h_{\mu\nu}}{v_h}, \tag{2.35}$$

where v_h has dimensions of mass and sets the scale of the metric fluctuations. We are interested in the case where the background metric is the Reissner-Nordstrom metric. However, for simplicity I shall expand around flat space instead. Because the arguments are essentially dimensional, they can be adapted to a positional space argument in a non flat background. We may then suppose that the low energy action of this theory comes as a functional of the fields $h_{\mu\nu}, A_\mu$ and their derivatives. Our action can written in the form:

[3] We could also expand in the Maxwell field into a background and fluctuations however it is not necessary

$$S_w = \int d^4x f^4 \sum_n c_n O_n(\frac{\partial_\mu}{M}, \frac{A_\mu}{v_A}, \frac{h_{\mu\nu}}{v_h}), \tag{2.36}$$

where f, v_A, v_h, M are all energy scales to ensure that the constants c_i are dimensionless and of order ≤ 1. We want to determine how the function $A_{\mathcal{E}_g\mathcal{E}_p}(p)$, representing Feynman graphs with $\mathcal{E}_g$ external gravitons and $\mathcal{E}_p$ external photons, all with energy $\sim p$, depends on these parameters. We denote $\mathcal{E}_g$ and $\mathcal{E}_p$ to be external graviton and photon lines respectively. I_g and I_p to be the number of internal graviton and photon lines respectively, with propagators $\propto \frac{1}{p^2}$ where p is the 4-momentum running through the line. V_n counts the number of vertices coming from a specific interaction in O_n. For each interaction term indexed by n, we have $f_{n,g}$ and $f_{n,p}$ graviton and photon fields converging at the corresponding vertex, and d_n denotes the number of derivatives. We state the identities which we shall need later.

$$2I_g + \mathcal{E}_g = \sum_n f_{n,g} V_n, \tag{2.37}$$

$$2I_p + \mathcal{E}_p = \sum_n f_{n,p} V_n. \tag{2.38}$$

We also define the number of loops L

$$L = 1 + I_g + I_p - \sum_n V_n. \tag{2.39}$$

We then use the Feynman rules to construct the graphs. One thing to be careful about is the fact that our Lagrangian is not canonically normalized.

$$(\text{Vertices}) = \prod_n \left[i(2\pi)^4 \delta^4(p) f^4 \left(\frac{p}{M}\right)^{d_n} \left(\frac{1}{v_A}\right)^{f_{n,p}} \left(\frac{1}{v_h}\right)^{f_{n,g}} \right]^{V_n} \tag{2.40}$$

$$(\text{Internal Graviton Line}) = \left[-i \int \frac{d^4p}{(2\pi)^4} \left(\frac{M^2 v_h^2}{f^4}\right) \frac{1}{p^2} \right]^{I_g} \tag{2.41}$$

$$(\text{Internal Photon Line}) = \left[-i \int \frac{d^4p}{(2\pi)^4} \left(\frac{M^2 v_p^2}{f^4}\right) \frac{1}{p^2} \right]^{I_p} \tag{2.42}$$

To proceed, we would like to give a dimensional estimate for the following integral where we associate p with the momenta through a loop

$$\int \cdots \int \left[\frac{d^4p}{(2\pi)^4}\right]^A \frac{p^B}{((p+q)^2)^C}. \tag{2.43}$$

To be clear, this is meaningless as an exact expression. Since we are only looking for a dimensionless estimate it is sufficient to label each loop momenta generically by p, much in the same way that we label all the external momentum generically by q, and essentially count powers of p in this integral. Of course this cannot capture all the intricacies of the full integral in a Feynman graph, such as potential singularities, IR divergences and non-analytic terms [7], however it is acceptable when considering the effect of the high energy behaviour. The most intuitive way to regularize this expression is by using a cutoff Λ as defined in section 2.1, which puts an upper limit on the energy running in loops. However, one drawback of this method is that Λ will appear in our final answer, even though it is not a physical quantity. In practise dimensional regularization is normally used because it does not introduce this artificial cutoff. In addition it also preserves symmetries such as Lorentz and gauge invariance. However, I will stick with using the cutoff because of its conceptual simplicity.

In the evaluation of the integral, the first thing we do is decompose d^4p into an angular and radial part. The angular part equals the volume of a 3-sphere, $2\pi^2$, therefore:

$$\int \cdots \int \left[\frac{d^4p}{(2\pi)^4}\right]^A \frac{p^B}{(p+q)^{2C}} \tag{2.44}$$

$$= \int \cdots \int \left[dpp^3 \frac{2\pi^2}{(2\pi)^4}\right]^A \frac{p^B}{(p+q)^{2C}} \tag{2.45}$$

$$= \left(\frac{1}{2(2\pi)^2}\right)^A \int \cdots \int dp^A \frac{p^{3A+B}}{(p+q)^{2C}} \tag{2.46}$$

We get another p^A factors from the integrals and evaluating at the cut off gives a dimensional estimate for this integral as:

$$\sim \left(\frac{1}{2(2\pi)^2}\right)^A \Lambda^{4A+B-2C}. \tag{2.47}$$

I have ignored any infrared divergences and dependence on q, because we are only interested in the highest energy behaviour. We now calculate the Feynman graphs for any process. We are interested in amputated graphs which are relevant to scattering amplitudes (via LSZ reduction etc). We can factor out one delta function which conserves external momentum. We then define $A_{\mathcal{E}_g\mathcal{E}_p}(q) = i(2\pi)^4\delta(q)\mathcal{A}_{\mathcal{E}_g\mathcal{E}_p}(q)$. The effect of the remaining momentum conserving delta functions is to remove one momentum integral each. Therefore the total integration remaining is $I - \sum_n V_n + 1 = L$. Instead of calculating $\mathcal{A}_{\mathcal{E}_g\mathcal{E}_p}(q)$ directly we wish to expand it as a power series: $\mathcal{A}_{\mathcal{E}_g\mathcal{E}_p}(q) = \mathcal{A}_{\mathcal{E}_g\mathcal{E}_p D}q^D$. If we think about about the functional form of $\mathcal{A}_{\mathcal{E}_g\mathcal{E}_p}(q)$, the external momentum are all present in the internal propagators (caused by the delta functions) $\frac{1}{(p+q)^2}$. This shows that each power of q will be divided by a p. Therefore our estimate for $\mathcal{A}_{\mathcal{E}_g\mathcal{E}_p D}q^D$ is:

$$\propto \int \cdots \int \left[\frac{d^4p}{(2\pi)^4}\right]^L \left(\frac{1}{p^2}\right)^{I_g+I_p} \left(\frac{q}{p}\right)^D \left(\frac{M^2v_h^2}{f^4}\right)^{I_g} \left(\frac{M^2v_p^2}{f^4}\right)^{I_p} \prod_n \left[f^4\left(\frac{p}{M}\right)^{d_n}\left(\frac{1}{v_A}\right)^{f_{n,p}}\left(\frac{1}{v_h}\right)^{f_{n,g}}\right]^{V_n} \tag{2.48}$$

$$\sim \left(\frac{1}{2(2\pi)}^2\right)^{2L} \Lambda^{4L-2(I_g+I_p)+\sum_n d_nV_n} \left(\frac{q}{\Lambda}\right)^D \left(\frac{M^2v_h^2}{f^4}\right)^{I_g} \left(\frac{M^2v_p^2}{f^4}\right)^{I_p} \prod_n \left[f^4\left(\frac{1}{v_A}\right)^{f_{n,p}}\left(\frac{1}{v_h}\right)^{f_{n,g}}\right]^{V_n} \tag{2.49}$$

This can be simplified greatly using the identities (2.37),(2.38) and (2.39). The following result is:

$$\mathcal{A}_{\mathcal{E}_g\mathcal{E}_p D}q^D \propto \left(\frac{1}{2(2\pi)^2}\right)^{2L} \left(\frac{q}{\Lambda}\right)^D \left(\frac{1}{v_h}\right)^{\mathcal{E}_g} \left(\frac{1}{v_p}\right)^{\mathcal{E}_p} \left(\frac{\Lambda}{M}\right)^{2L+\sum_n(d_n-2)V_n+2} \left(\frac{M\Lambda}{4\pi f^2}\right)^{2L} \tag{2.50}$$

There is much to say about this formula. Consider a process with fixed external momenta. As long as $f^2 \geq M\Lambda$, then the answer is suppressed for more complicated graphs. In each term the numerator is bigger (or equal the denominator). Therefore the only possible term which could cause enhancement of the answer for more complicated diagrams is $\left(\frac{\Lambda}{M}\right)^{2L+\sum_n(d_n-2)V_n+2}$, if we could choose vertices with 0 derivatives. However this is not possible. The minimum amount of derivatives in gravity interaction is 2 coming from the Einstein-Hilbert term. Likewise, due to gauge invariance each photon field comes with a derivative. Interaction terms must have 3 derivatives of more (the usual kinetic term has 2). Another interesting point is that for graviton scattering the least suppressed terms correspond to the diagrams with $L = 2$ and $\sum_n (d_n - 2)V_n + 2 = 0$. This corresponds to tree diagrams with any number of interaction vertices coming from the Einstein-Hilbert term. This is what we would expect as these are the diagrams coming from normal GR.

For photon-photon scattering diagrams, it is not possible to choose $\sum_n (d_n - 2)V_n + 2 = 0$, as a argued above. Therefore those processes are suppressed at least by some powers of M. Again this is exactly what we expect. In QED photons do not self interact. Therefore the photon scattering that we see in the low energy limit comes from diagrams mediated by heavy particles (e.g the electron) and is suppressed.

Finally, suppose we are looking at a specific process with $\mathcal{E}_g$ external gravitons and $\mathcal{E}_p$ external photons with the number of loops and vertices unconstrained. We see that scale which appears in the fields does not appear in any terms apart from those governed by the number of external lines. Thus it is consistent to have the field value take a large value without ruining our expansion. Specific to this case, we may take the scale $v_A \sim O(1)$ without ruining our expansion. This is not unreasonable if we consider the photon field for a Reissner-Nordstrom Black hole: $A_\mu = (0, -\frac{M_{pl} r_Q}{r}, 0, 0)$. For an extremal black hole at the horizon, $r = r_s = |r_Q|$ and therefore $A_r \sim M_{pl}$[4].

2.3.1 Applicability to Black Holes

Since black holes are some of the most bizarre and non-intuitive objects we encounter in physics it is worth spending some time discussing whether the effective field theory framework is still valid. It is clear that our framework cannot be used to describe behaviour near the singularity. We are instead particularly interested in behaviour near the horizon. It is argued in [8] that for large black holes, i.e $M_{BH} \gg M_{pl}$, we expect this framework to remain valid. For large black holes, the curvature at the horizon is small so there is no reason to believe that the effective field theory regime

[4] r_s, r_Q are defined in section 3.1. The normalisation of A_μ in this convention is also given there

breaks down there.

Concerns have also been raised whether it is possible to have a fully consistent low energy effective near the horizon. For example static observers near the horizon will see extremely high energy modes. It is again argued in [8] that this is not a problem. Consider a low-energy scattering process in flat space. By performing a Lorentz transformation, it is certainly possible to choose a frame in which the scattering process no longer appears to be low-energy. However, we know that physics doesn't depend on the frame. The essential point is that the frame invariant quantities entering the calculation need to be describable using a low energy theory. Another way to look at this is that if there exists a frame such that the process can be described by low energy physics, then it can validly be described by a low energy effective field theory. Indeed this is true for black holes: we can choose a nice foliation of spacetime, in which we only encounter low energy behaviour. There are some additional technicalities involving Hawking radiation and I would encourage the reader to look at [8, 27] if interested.

Chapter 3

Classical Reissner-Nordstrom Black Hole

In this chapter I shall examine the solutions to electrically charged black holes. I shall begin by considering the Einstein-Maxwell action in regular General Relativity and show a derivation of the Reissner-Nordstrom solution from this action. Then, I will consider the effective action (2.38) and look for slight deviations to Reissner-Nordstrom solution caused by the addition of the suppressed $R_{\mu\nu}R^{\mu\nu}$ term.

3.1 Reissner-Nordstrom Metric

The classical action for a charged black hole is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{M_{pl}^2}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right], \tag{3.1}$$

where $F_{\mu\nu}$ is the field strength tensor defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ from the Maxwell field and the R is the Ricci scalar. We can vary this action with respect to the metric to find Einstein's equations for the charged black hole:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad T_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \tag{3.2}$$

We have used the convention $M_{pl}^2 = \frac{1}{8\pi G}$ to put in typical form. $T_{\mu\nu}$ is traceless and thus $R = 0$ so we can simplify this equation

$$R_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{3.3}$$

We have two further conditions coming from Maxwell's equations in curved space-time

$$\nabla_\mu F^{\mu\nu} = 0 \qquad\qquad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{3.4}$$

There are a few different methods that can be used to derive the Reissner-Nordström metric from this action. The classic way is to first vary the action to derive Einstein's field equations as I have done above. Then we can substitute an ansatz for spherically symmetric solutions into the field equations and then solve the resulting equations to find the exact form of the metric. This is perfectly acceptable, however it is rather algebraically complicated. Instead I will use a method initially discovered by Weyl [5] and extended by Deser and Franklin [6]. The general idea is to first restrict our field variables to their most general form allowed by symmetries, which is in our case spherical symmetry, then re-insert these into the action and then vary these variables and solve the resulting equations.

There is some question whether this method gets all the solutions however for the purpose of this work it is sufficient. One advantage of using this method is that it is similar to the method that I will use to calculate the perturbed metric when I include the effect of higher dimensional operators.

We now proceed to the calculation. Due to spherical symmetry we can restrict the form of the metric and the 4-potential to:

$$ds^2 = a(r,t)dt^2 + b(r,t)dr^2 + c(r,t)dtdr + e(r,t)r^2 d\Omega^2 \tag{3.5}$$

$$A_0 = A_0(r,t) \quad A_r = A_r(r,t) \quad A_\theta = A_\phi = 0 \tag{3.6}$$

We can then use coordinate freedom $r \to r - \frac{c}{b}t$ to set the $drdt$ term to zero. Likewise we can use U(1) gauge invariance in order to set $A_r = 0$. (The Coloumb gauge. Finally, we scale r to set $e = 1$ to put the metric in the form.

$$ds^2 = ab^2 dt^2 + \frac{1}{a}dr^2 + r^2 d\Omega^2 \tag{3.7}$$

In [6] the $drdt$ term is not gauged away and is necessary to prove Birkhoff's theorem. We then calculate the Ricci Scalar and Field Strength tensor and insert back into the action

$$R = \frac{1}{r^2 b}(2b - 2ab - 4rba' - 4rab' - 3r^2 a'b' - r^2 ba'' - 2r^2 ab''), \tag{3.8}$$

$$F_{\mu\nu}F^{\mu\nu} = \frac{2}{b^2}A_0'^{\,2}, \tag{3.9}$$

$$S_{red} = 4\pi \int drdt \;\; \frac{M_{pl}^2}{2}(2b - 2ab - 4rba' - 4rab' - 3r^2 a'b' - r^2 ba'' - 2r^2 ab'') + \frac{1}{2b}r^2 A_0'^2, \tag{3.10}$$

where the primes indicate differentiation with respect to r. After integrating by parts and removing the boundary terms this can be put in the rather more simple form

$$S_{red} = 4\pi \int drdt \;\; \frac{M_{pl}^2}{2}(-2b(ar - r)') + \frac{1}{2b}r^2 (A_0)'^2 \tag{3.11}$$

Varying the action with respect to these variables and setting its result equal zero

$$\frac{\delta S_{red}}{\delta A_0} = 0 \rightarrow \partial_r(\frac{r^2 A_0'}{b}) = 0 \tag{3.12}$$

$$\frac{\delta S_{red}}{\delta a} = 0 \rightarrow \partial_r b = 0 \tag{3.13}$$

$$\frac{\delta S_{red}}{\delta b} = 0 \rightarrow \frac{M_{pl}^2}{2}(1 - a - a'r) + \frac{1}{4b^2}r^2(A_0')^2 = 0 \tag{3.14}$$

The solution of these equations can be simply solved as follows:

$$b = constant \tag{3.15}$$

$$A_0 = -\sqrt{2}M_{pl}\frac{r_Q}{r} \tag{3.16}$$

$$a = 1 - \frac{r_s}{r} + \frac{r_Q^2}{b^2 r^2} \tag{3.17}$$

where r_s is the Schwarzchild radius and r_Q is the characteristic length scale associated to the charge. We can perform one final coordinate redefinition $t \to t'$ where $bdt = dt'$ in order to set $b \to 1$. Therefore we find the familiar form of the Reissner-Nordstrom metric:

$$ds^2 = -(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2})dt^2 + (1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2})^{-1}dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 \quad (3.18)$$

It is also useful to determine the location of any event horizons and singularities. There are coordinate singularities, which can be associated to event horizons with a bit more work, at

$$r_+ = \frac{1}{2}\left(r_s + \sqrt{r_s^2 - 4r_Q^2}\right), \quad (3.19)$$

$$r_- = \frac{1}{2}\left(r_s - \sqrt{r_s^2 - 4r_Q^2}\right), \quad (3.20)$$

$$(3.21)$$

and a curvature singularity at

$$r = 0 \quad (3.22)$$

We can distinguish between the geometry in three parametric ranges: $r_s < |2r_Q|, r_s = |2r_Q|, r_s > |2r_Q|$.

$\mathbf{r_s < |2r_Q|}$

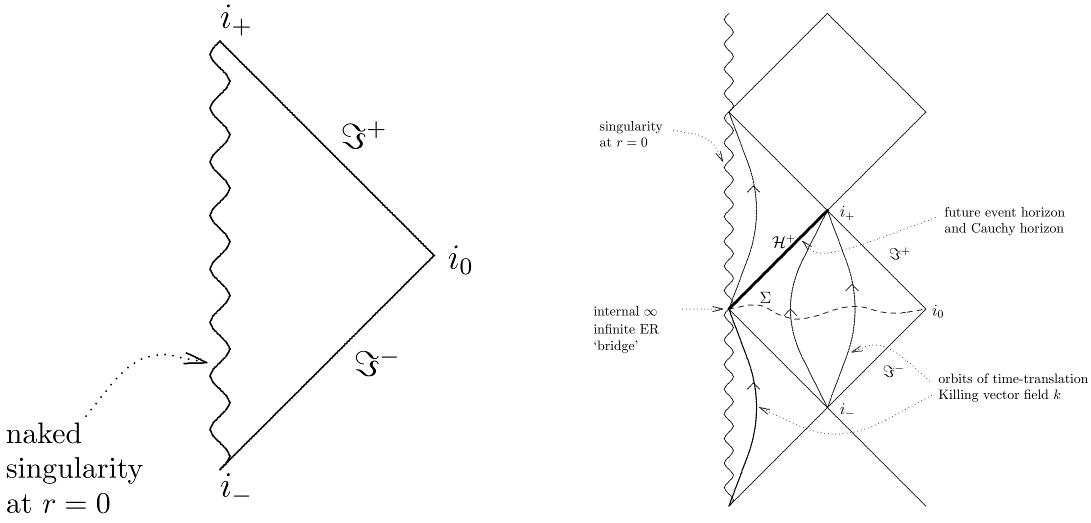
In this range $r_\pm$ are both imaginary and therefore there is no event horizon. The singularity at $r = 0$ is known as a naked singularity.

$\mathbf{r_s = |2r_Q|}$

In this range $r_\pm = \frac{1}{2}r_s$, therefore there is a single event horizon. This type of black hole is known as an extremal black hole.

$\mathbf{r_s > |2r_Q|}$

In this range, $r_\pm$ are both real and different, therefore there are two event horizons.



(a) $r_s < 2r_Q$

(b) $r_s = 2r_Q$

Figure 3.1: The maximally extended Penrose Diagram for two different geometries. The diagrams are taken from [28]

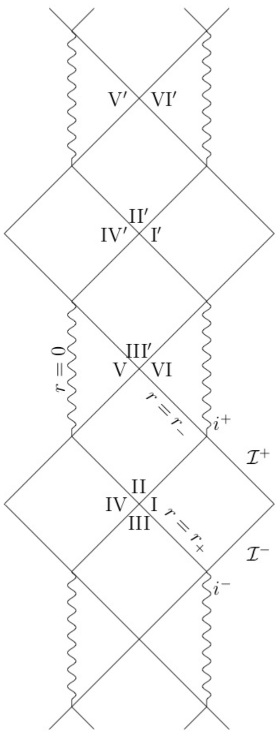


Figure 3.2: The maximally extended Penrose Diagram for $r_s > 2r_Q$. Diagram taken from [29]

3.2 Higher Dimension Operators

In this section, we extend the method developed in section 3.1 to look for deviations of the Reissner-Nordstrom metric coming from the higher dimensional operators in the Lagrangian. The leading order corrections are caused by the $R_{\mu\nu}R^{\mu\nu}$ term and therefore deviations from the Reissner-Nordstrom geometry should be suppressed by the dimensionless parameter[1]

$$\epsilon = \frac{1}{M_{pl}^2 r_+^2} \tag{3.23}$$

Since our interest is in static and spherically symmetric solutions we make the following ansatz for the metric and Maxwell field:

$$ds^2 = -A(r)dt^2 + \frac{1}{B(r)^2}dr^2 + C(r)r^2(d\theta^2 + sin^2\theta) \tag{3.24}$$

$$A_0 = A_0(r) \quad A_r = A_r(r) \quad A_\theta = A_\phi = 0 \tag{3.25}$$

We can then do a coordinate redefinition $r \to \frac{1}{\sqrt{C(r)}}r$, in order to set $C(r) = 1$. We can also gauge away the A_r term as in the previous section. We substitute this ansatz into the Lagrangian and vary with respect to $A(r), B(r)$ and $A_0(r)$, setting each variation to zero, resulting in three equations $\mathcal{E}_A, \mathcal{E}_B$ and $\mathcal{E}_{A_0}$. Our aim is to then use these three equations in order to solve A, B and A_0 perturbatively to first order in ϵ. Since we expect deviations from the Reissner-Nordström geometry to be suppressed by ϵ, we can make the following ansatz for A, B and A_0

$$A(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} + \epsilon a(r) \tag{3.26}$$

$$B(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} + \epsilon b(r) \tag{3.27}$$

$$A_0(r) = -\sqrt{2}M_{pl}(\frac{r_Q}{r} + \epsilon a_0(r)) \tag{3.28}$$

$$\tag{3.29}$$

[1]The use of r_+ as the length scale is because we shall be generally interested in behaviour on the outer event horizon. This is because the inner event horizon is plagued by instabilities.

The resulting equations can be solved using Mathematica to obtain solutions.

$$A(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} + \epsilon \left[\frac{cr_+^2 r_Q^2(-2r_Q^2 + 5r(r_s - 2r)}{5r^6} \right], \tag{3.30}$$

$$B(r) = 1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2} + \epsilon \left[\frac{cr_+^2 r_Q^2(-12r_Q^2 + 5r(3r_s - 4r))}{5r^6} \right], \tag{3.31}$$

$$A_0(r) = -\sqrt{2}M_{pl}^2 \left(\frac{q}{r} + \epsilon \left[\frac{cr_+^2 r_Q^3}{5r^5} \right] \right). \tag{3.32}$$

Higher dimensional operators cannot lead to any physical singularity within in the region of validity of the EFT. Since our higher dimensional operators are suppressed by a small constant, we expect departures from the Reissner-Nordstrom geometry to also be small. The creation of a physical singularity is a radical change in geometry, associated with high energy processes, and would likely indicate that the effective field theory is no longer valid. In order to enforce this A and B must vanish simultaneously $A(r = r_h) = B(r = r_h) = 0 + O(\epsilon^2)$ at the same point defining an event horizon. It is possible that neither of these functions vanish at any point and thus the theory has no event horizon. Alternatively, it is possible that they vanish simultaneously at multiple points defining multiple horizons. The classic way to find a singularity is by evaluating a scalar quantity and seeing if it diverges. Evaluating the Kretchmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, we see that if A(r) and B(r) do not vanish at the same point then there is a physical singularity which is not allowed. Again, the geometry of the black hole will depend on the ratio of r_s to r_Q. These will approximately follow the same parametric ranges as the classical case, but we expect deviations to these of $O(\epsilon)$. The condition for a black hole to be extremal is found to be

$$r_s^2 = 4r_Q^2 - \epsilon \frac{8}{5} cr_+^2. \tag{3.33}$$

I shall now give the location of the perturbed horizons, if any exist, in the three parametric ranges:

$\mathbf{r_s^2 < 4r_Q^2} - \epsilon\mathbf{\frac{8}{5}cr_+^2}$

There is no event horizon, as we would expect from the classical result

$\mathbf{r_s^2 = 4r_Q^2} - \epsilon\mathbf{\frac{8}{5}cr_+^2}$

There is an event horizon at $r = \frac{1}{2}r_s$.

$\mathbf{r_s^2 > 4r_Q^2} - \epsilon\mathbf{\frac{8}{5}cr_+^2}$

There are two event horizons at

$$r_{p,+} = r_+ + \epsilon\frac{cr_Q^2(5r_+^2 - 3r_Q^2)}{5r_+(r_+^2 - r_Q^2)}, \tag{3.34}$$

$$r_{p,-} = r_- + \epsilon\frac{cr_+^2 r_Q^2(5r_-^2 - 3r_Q^2)}{5r_-^3(r_-^2 - r_Q^2)}, \tag{3.35}$$

where $r_\pm$ are the classical event horizons defined earlier

Chapter 4

Fluctuations on a EFT-RN background

In this chapter, I will analyse metric and electromagnetic perturbations on top of Reissner-Nordstrom background solutions. I shall do this first for perturbations on top of a classical Reissner-Nordstrom background in pure GR and then on top of the perturbed background solution found in section 3.2 for the EFT. The strategy is to firstly find the form of metric and electromagnetic perturbations in a suitable gauge. Then plug these into the linearized equations of motions to find the dynamics of the physical modes.

4.1 EFT equations of motion

The equations of motion coming from the variation of metric in the action (2.34) can be written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{1}{M_{pl}^2}(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}) + \frac{2}{M_{pl}^2}\mathcal{E}_{\mu\nu} = 0, \tag{4.1}$$

where $\mathcal{E}_{\mu\nu}$ are the terms coming from the higher dimensional operators. Even though the Ricci scalar is zero for Reissner-Nordstrom solution, there is no reason that the perturbations of it will also be zero, therefore we should keep it in our equations of motion. The form of $\mathcal{E}_{\mu\nu}$ is derived in appendix and is given by:

$$\mathcal{E}_{\mu\nu} = c(2R_{\mu}{}^{\sigma}R_{\nu\sigma} - \frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} - 2\nabla_{\alpha}\nabla_{\nu}R_{\mu}{}^{\alpha} + \frac{1}{2}g_{\mu\nu}\Box R + \Box R_{\mu\nu}). \tag{4.2}$$

Since there are no higher dimensional operators containing $F_{\mu\nu}$ in the Lagrangian we are considering, Maxwell's equations remain unchanged.

4.2 Metric and Electromagnetic perturbations

We expand the metric g and Maxwell field A around background solutions $\bar{g}$ and $\bar{A}$ such that:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{4.3}$$

$$A_{\mu} = \bar{A}_{\mu} + a_{\mu}. \tag{4.4}$$

Having done this we may define the perturbed Maxwell tensor $f_{\mu\nu}$ by $\partial_{\mu}a_{\nu} - \partial_{\nu}a_{\mu}$. We can then expand the Maxwell tensor around it background solution

$$F_{\mu\nu} = \bar{F}_{\mu\nu} + f_{\mu\nu}. \tag{4.5}$$

The specific form of the background solution depends on which theory we are considering- GR or the EFT. For both cases, the perturbations take the same form. The spherical symmetry of the background allows us to simplify this problem greatly: we can split the $h_{\mu\nu}$ and $f_{\mu\nu}$ into a part which depends on angular variables (θ, ϕ) and a part which depends on (t, r). Additionally, we may only consider axisymmetric modes, m $= 0$, because non-axisymmetric modes can be deduced via rotations due to spherical symmetry.

Similarly to the analysis of metric perturbations in flat space, we have a redundancy in our description of $h_{\mu\nu}$. Infinitesimal coordinate transformations present themselves as gauge transformations, and using these we can remove redundancies. In flat space we impose the de Donder and then the transverse traceless gauge in order to reduce $h_{\mu\nu}$ to two degrees of freedom. In this case, the most convenient gauge choice is the Regge-Wheeler gauge condition [1] in which perturbations are written as $h_{\mu\nu} = h^o_{\mu\nu} + h^e_{\mu\nu}$ where:

$$h^o_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0 \\ 0 & 0 & 0 & h_1 \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{pmatrix} sin(\theta) Y_l^{'}(\theta), \tag{4.6}$$

and

$$h^e_{\mu\nu} = \begin{pmatrix} AH_0 & H_1 & 0 & 0 \\ H_1 & H_2/B & 0 & 0 \\ 0 & 0 & r^2 K & 0 \\ 0 & 0 & 0 & r^2 sin^2(\theta) K \end{pmatrix} Y_l(\theta), \tag{4.7}$$

where $h_0, h_1, H_0, H_1, H_2, K$ are all functions of t and r. $Y_l(\theta) = Y_{l0}(\theta, \phi)$ is the usual spherical harmonic and the prime denotes differentiation by θ. We are only interested in modes with $l \geq 2$ because the others do correspond to radiative modes[1]. A and B are both functions of r corresponding to (3.17) for GR and (3.26),(3.27) for the EFT respectively.

The reason that we distinguish between $h^o_{\mu\nu}$, called axial or odd perturbations, and $h^e_{\mu\nu}$, called polar or even perturbations, is because they transform differently under a parity transformation: $(\theta, \phi) \to (\pi - \theta, \phi + \pi)$. The reason this is relevant is that Reissner-Nordstrom metric is parity invariant and therefore to linear order the even and odd perturbations do not couple and can be treated separately.

In Regge-Wheeler gauge we have 6 independant degrees of freedom rather than the 2 derived in the transverse traceless gauge in flat space. The number of degrees of freedom certainly has not changed. Rather, we still have additional gauge freedom after choosing the Regge-Wheeler gauge. This additional gauge freedom will manifest in an overdetermined system of equations and we should only be left with two independant equations for an odd and even mode.

The electromagnetic perturbations $f_{\mu\nu}$ can be deduced from vector spherical harmonics. We could also do by noticing that an anti-symmetric tensor, which is an irreducible representation of the Lorentz group, decomposes into two vector

[1] From now on everything I say is for $l \geq 2$

representations under the rotation group, as long as we are careful about maintaining U(1) gauge invariance. Similarly to the metric case, these perturbations can be split into their odd and even part. We use the basis given by Zerilli [2] where:

$$f^{o}_{\mu\nu} = M_{pl}\begin{pmatrix} 0 & 0 & 0 & -\tilde{f}_{02} sin\theta Y_l^{'}(\theta) \\ 0 & 0 & 0 & -\tilde{f}_{12} sin\theta Y_l^{'}(\theta) \\ 0 & 0 & 0 & \tilde{f}_{23} sin\theta Y_l(\theta) \\ \tilde{f}_{02} sin\theta Y_l^{'}(\theta) & \tilde{f}_{12} sin\theta Y_l^{'}(\theta) & -\tilde{f}_{23} sin\theta Y_l(\theta) & 0 \end{pmatrix}, \tag{4.8}$$

and

$$f^{e}_{\mu\nu} = M_{pl}\begin{pmatrix} 0 & \tilde{f'}_{01} Y_l(\theta) & \tilde{f'}_{02} Y_l^{'}(\theta) & 0 \\ -\tilde{f'}_{01} Y_l(\theta) & 0 & \tilde{f'}_{12} Y_1^{'}(\theta) & 0 \\ -\tilde{f'}_{02} Y_l^{'}(\theta) & -\tilde{f'}_{12} Y_1^{'}(\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.9}$$

w,here $\tilde{f}_{\mu\nu}$[2] are functions of t and r and as before $Y_l(\theta) = Y_{l0}(\theta, \phi)$ are the spherical harmonics with m=0. Again this is important because to linear order, only odd/even electromagnetic perturbations will couple with odd/even gravitational perturbations. These components given are actually not all independant: Maxwell's equations (1.64) can be used to write the following relations:

$$\bar{f}_{12} = \frac{1}{l(l+1)}\frac{\partial \bar{f}_{23}}{\partial_r} \tag{4.10}$$

$$\bar{f}_{02} = \frac{1}{l(l+1)}\frac{\partial \bar{f}_{23}}{\partial_t} \tag{4.11}$$

$$\bar{f'}_{01} = \frac{\partial \bar{f'}_{02}}{\partial_r} - \frac{\partial \bar{f'}_{12}}{\partial_t} \tag{4.12}$$

4.3 Linearized Einstein-Maxwell equations

We would like to proceed to determining the dynamics of these modes. The typical way this is done is by first linearizing the equations of motion. By which I mean

[2]The M_{pl} factor in this definition is rather unconventional, however we have chosen it to be in line with the normalisation of the background (3.28). This stops there from being M_{pl} dotted around our zeroth order equations. Also note that the $\tilde{f}_{\mu\nu}$ do not all have the mass dimension. i.e $[\tilde{f}_{23}] = 1$ but $[\tilde{f}_{12}] = 0$

write them to first order in the fluctuation around the background. We may plug in the form of the perturbations to find the dynamics of the modes.

4.3.1 Linearized Equations in General Relativity

The linearized Einstein-Maxwell equations in General Relativity are given by:

$$G^{(1)}_{\mu\nu} = 8\pi G \left(T^{(1,h)}_{\mu\nu} + T^{(1,f)}_{\mu\nu}\right), \tag{4.13}$$

where

$$G^{(1)}_{\mu\nu} = -\frac{1}{2}\Big[\Box h_{\mu\nu} + \bar{\nabla}_\mu \bar{\nabla}_\nu h + 2R^{\lambda}{}_{\mu}{}^{\alpha}{}_{\nu} h^{\lambda\alpha} - \bar{\nabla}_\mu \bar{\nabla}_\alpha h_\nu{}^\alpha - \bar{\nabla}_\nu \bar{\nabla}_\alpha h_\mu{}^\alpha \tag{4.14}$$
$$- \bar{R}^{\lambda}{}_{\mu} h_{\nu\lambda} - \bar{R}^{\lambda}{}_{\nu} h_{\mu\lambda} - \frac{1}{2}\bar{g}_{\mu\nu}(\bar{\nabla}^\alpha \bar{\nabla}^\beta h_{\alpha\beta} - \Box h - \bar{R}^{\alpha\beta} h_{\alpha\beta})\Big],$$

$$T^{(1,h)}_{\mu\nu} = -\Big[\bar{g}^{\alpha\sigma}\bar{g}^{\rho\gamma}(\bar{F}_{\mu\rho}\bar{F}_{\nu\alpha} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\beta\lambda}\bar{F}_{\alpha\beta}\bar{F}_{\rho\lambda})h_{\sigma\gamma} + \frac{1}{4}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta}h_{\mu\nu}\Big], \tag{4.15}$$

$$T^{(1,f)}_{\mu\nu} = \bar{g}^{\alpha\rho}(\bar{F}_{\mu\rho}f_{\nu\alpha} + \bar{F}_{\nu\rho}f_{\mu\alpha}) - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\gamma\alpha}\bar{g}^{\beta\delta}f_{\alpha\beta}\bar{F}_{\gamma\delta}, \tag{4.16}$$

and

$$\frac{\partial}{\partial\mu}(\frac{1}{2}h\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}F_{\alpha\beta} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}f_{\alpha\beta} - \bar{g}^{\mu\alpha}h^{\nu\beta}F_{\alpha\beta} - \bar{g}^{\nu\beta}h^{\mu\alpha}F_{\alpha\beta}) = 0, \tag{4.17}$$

where $h = g^{\mu\nu}h_{\mu\nu}$

4.3.2 Linearized Equations in the EFT

We could also linearize the additional terms in the equations coming from the higher dimensional operator in the EFT (4.2). However deriving these by hand is quite tedious. In fact, we can skip this step entirely by plugging the background solution and perturbations into the full equations of motion and only looking at the terms linear in perturbations. This is easiest done using a symbolic manipulator program such as Mathematica.

4.4 Master Equations for Odd Modes

In this section, I will give the master equations for the dynamics of the odd modes. As I mentioned previously, although the Regge-Wheeler gauge has 6 degrees of freedom, additional gauge freedom means there is only actually only 2 degrees of freedom which appear as one odd and one even mode. In General Relativity, the equations determining the dynamics of these modes on a Reissner Nordstrom background, are known as the Zerilli equations.

4.4.1 Zerilli Equations

In the odd sector we obtain three equations from the Einstein's equations

$$h_0(-2+J+2A+2rA'+r^2A'')+rA(-r\partial_r^2 h_0+2\partial_t h_1+r\partial_t\partial_r h_1)=\frac{2r_Q(r_Q h_0+\sqrt{2}r^2A\bar{f}_{12})}{r^2}, \tag{4.18}$$

$$Ah_1(-2+J+2rA'+r^2A'')+r(2\partial_t h_0+r(-\partial_t\partial_r h_0+\partial_t^2 h_1))=\frac{2r_Q(r_Q A h_1+\sqrt{2}r^2\bar{f}_{02})}{r^2}, \tag{4.19}$$

$$AA'h_1+A^2\partial_r h_1-\partial_t h_0=0, \tag{4.20}$$

and one equation from Maxwell's equation

$$A\partial_r f_{12}+\frac{2\sqrt{2}r_Q h_0+r(\bar{f}_{23}-r^2\bar{f}_{12}A'-\sqrt{2}r_Q\partial_r h_0+\sqrt{2}r_Q\partial_t h_1)}{r^3}+\frac{\partial_t\bar{f}_{02}}{A}=0, \tag{4.21}$$

where primes denote differentiation by r and $J=l(l+1)$. I shall also perform a Fourier transform of $h_i(r,t)$ and $\bar{f}_{\mu\nu}(r,t)$ with respect to the time variable

$$h_i(t,r)=\int_{\infty}^{\infty}\frac{dw}{2\pi}\tilde{h}_i(w,r)e^{-iwt}. \tag{4.22}$$

Then we can use (4.20) to write $\tilde{h}_0$ in terms of $\tilde{h}_1$. Inserting this back into (4.19), we can write as a Schrodinger type equation:

$$\partial_*^2\psi^o + w^2\psi^o - \frac{A\psi^o}{r^2}(J - \frac{3r_s}{r} + \frac{4r_Q^2}{r^2}) = \frac{-2\sqrt{2}iwAr_Q}{r^3}f_{LM} \tag{4.23}$$

Where we have defined:

$$\psi^o(w,r) = \frac{A\tilde{h_1(w,r)}}{r}, \tag{4.24}$$

$$f_{LM} = \frac{1}{J}f_{23}, \tag{4.25}$$

$$\partial_* = A\partial_r. \tag{4.26}$$

We can also put the Maxwell equation into a similar form. Again, we use (4.20) to substitute $\tilde{h}_0$ for $\tilde{h}_1$. We also need to use (4.23) to finally put in the form:

$$\partial_*^2 f_{LM} + w^2 f_{LM} - \frac{Af_{LM}}{r^2}(J + \frac{4r_Q^2}{r^2}) = \frac{-i\sqrt{2}Ar_Q(J-2)}{wr^3}\psi^o \tag{4.27}$$

These are two coupled differential equation however it is not too hard to uncouple them. We can write in matrix form as:

$$(\partial_*^2 + w^2 - \frac{A}{r^2}(J + \frac{4r_Q^2}{r^2}))\begin{pmatrix}\psi^o \\ wf_{LM}\end{pmatrix} + \frac{\sqrt{2}A}{r^3}\begin{pmatrix}3r_s & 2ir_Q \\ ir_Q(J-2) & 0\end{pmatrix}\begin{pmatrix}\psi^o \\ wf_{LM}\end{pmatrix} = 0 \tag{4.28}$$

It is a simple task to find the eigenvectors of this matrix, which then is used to perform a redefinition of ψ^o and f_{LM} which decouple these equations.

4.5 Modified Zerilli Equations in EFT

In this section we now include the effects from the higher dimensional operators. We again get three equations from Einsteins equations and one from Maxwell's equation. Unsurprisingly the higher dimensional operators leads to higher derivative terms, however these are not a problem and can be perturbatively removed using the lower order of equations using the procedure outlined in [12]. Similarly to GR case, this leads to a pair of coupled differential equations involving the two modes, h_1, f_{lm}, of the form:

$$a_1\frac{\partial^2 h_1}{\partial_r^2}+a_2\frac{\partial h_1}{\partial_r}+a_3w^2h_1+\epsilon(a_4h_1+a_5\frac{\partial h_1}{\partial_r}+a_6wf_{lm}+a_7w\frac{\partial f_{lm}}{\partial_r})=a_8wf_{lm}, \tag{4.29}$$

$$b_1\frac{\partial^2 f_{lm}}{\partial_r^2}+b_2\frac{\partial f_{lm}}{\partial_r}+b_3w^2f_{lm}+\epsilon(\frac{b_4}{w}h_1+\frac{b_5}{w}\frac{\partial h_1}{\partial_r}+b_6f_{lm}+b_7\frac{\partial f_{lm}}{\partial_r})=b_8\frac{h_1}{w}, \tag{4.30}$$

where the a_i and b_i are functions of r. These are such that they return the Zerilli equation to zeroth order in ϵ. We have also fourier transformed h_i as in (4.22) The steps used to derive these equations are discussed in detail in appendix. Unfortunately, it is not clear how to proceed from this point, specifically how to decouple these two equations and remove both the $\frac{\partial f_{lm}}{\partial_r}$ and $\frac{\partial h_1}{\partial_r}$. Presumably they may be removed via some redefinition f_{lm} and h_1 however is immediately obvious how this may be done. However, all is not lost. We may make the simplification of setting electromagnetic perturbations to zero. In this limit, we may only consider a reduced form of (4.29)

$$a_1\frac{\partial^2 h_1}{\partial_r^2}+a_2\frac{\partial h_1}{\partial_r}+a_3w^2h_1+\epsilon(a_4h_1)+a_5\frac{\partial h_1}{\partial_r}=0. \tag{4.31}$$

We can put this is a more convenient form

$$\partial_*^2\psi^o+w^2\psi^o-\frac{\sqrt{AB}\psi^o}{r^2}(V_{GR}+\epsilon V_{HDO})=0, \tag{4.32}$$

where we use the definitions $\partial_*=\sqrt{AB}\partial_r$ and $\psi^o=\frac{\sqrt{AB}h_1}{r}[1+\epsilon f_{h_1}]$. The details of the potential are left for the appendix. Looking at this equation it is clear to see that there is no correction to the low-energy radial speed. I shall comment more on this in the conclusion.

Chapter 5

Conclusion and Further Development

In this paper we have investigated the propagation of gravitational waves on a Reissner-Nordstrom black hole background in low energy EFT of gravity and electromagnetism. The motivation for this was provided by work which showed that the low energy speed of photons may deviate from unity in a background gravitational field [10, 17–26] and particularly [11], which showed that speed of low energy gravitational waves propagating on a Schwarzchild background in an EFT of gravity deviated from unity.

We have considered a low energy EFT of gravity and electromagnetism including operators which have up to four derivatives in the light fields $g_{\mu\nu}$ and A_μ. We find that the leading order corrections to action coming from these four derivative operators can be reduced to only three terms. Out of these three terms, we have chosen to disregard two which would cause the Maxwell field to no longer be minimally coupled. This is to make sure that the speed of photon is unity (as compared to its value in GR). Therefore the leading order correction comes from a $R_{\mu\nu}R^{\mu\nu}$ term. In the limit in which electromagnetic perturbations are zero, we have derived a modified Zerilli equation for the odd gravitational modes in the EFT. We do not find a departure of the radial speed of gravitational waves from unity as compared to the speed of photon.

More analysis is required to understand why this is the case. Perhaps there is some unforeseen identity involving $R_{\mu\nu}R^{\mu\nu}$ that makes it irrelevant. Additionally, it is seen that the speed of photons propagating radially on a classical Reissner Nordstrom background in a low energy effective field theory of Electromagnetism is not modified. [24]. Perhaps the coupling of these two modes in the perturbation equations ensure their speeds stay equal.

We would like to briefly connect with phenomenology. Obviously, we find no modification to the speed of gravitational waves compared to General Relativity for this particular Lagrangian. However it is worthwhile to look at the scale of deviations we would expected if there were any. The suppression for the derivative four terms (near the outer event horizon) is given by $\epsilon = \frac{1}{M_{pl}^2 r_+^2}$. This is approximately [1]:

$$\epsilon \sim \frac{M_{pl}^2}{M_{BH}^2}, \tag{5.1}$$

For astrophysical black holes, this is observationally tiny so we should not expect analysis like these to be used for physical predictions.

There a few avenues for further development. The most obvious is to re-include the effects from the RF^2 and F^4 type operators. It would be interesting to see whether these operators would have an effect on the speed of the gravitational waves. Another interesting extension would be to explore the propagation of gravitational waves on a background Kerr metric. Further afield, it would be interesting to repeat the power counting arguments on the the background of the black hole rather than flat space.

[1] Astrophysical black holes have very little net charge so $r_+ \sim r_s$. This only changes the result by at most a factor of 2 anyway which turns out to be unimportant

Appendix A

Appendix

A.1 Linearized Einstein-Maxwell in General Relativity

Here we present a derivation of the Einstein-Maxwell equations to linear order in perturbations. We shall first consider the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}R = 8\pi G T_{\mu\nu} \tag{A.1}$$

We shall first look at the curvature terms on the left hand side of this. We begin with the form linearized Ricci Tensor derived in [31]

$$R^{(1)}_{\mu\nu} = \frac{1}{2}(\bar{\nabla}^\alpha \bar{\nabla}_\mu h_{\nu\alpha} + \bar{\nabla}^\alpha \bar{\nabla}_\nu h_{\mu\alpha} - \bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\mu\nu} - \bar{\nabla}_\nu \bar{\nabla}_\mu h) \tag{A.2}$$

Contracting with the metric, we obtain the linearized Ricci Scalar.

$$R = g^{\mu\nu} R_{\mu\nu} = (\bar{g}^{\mu\nu} - h^{\mu\nu})(\bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu}) = -h^{\mu\nu}\bar{R}_{\mu\nu} + \bar{\nabla}^\alpha \bar{\nabla}^\beta h_{\alpha\beta} - \Box h \tag{A.3}$$

We can use an identity relating the commutation of covariant derivatives to the Riemann tensor

$$(\bar{\nabla}_\alpha \bar{\nabla}_\mu - \bar{\nabla}_\mu \bar{\nabla}_\alpha) h_\nu{}^\alpha = R_{\lambda\mu} h_\nu{}^\lambda - R^\lambda{}_{\nu\alpha\mu} h_{\lambda\alpha}, \tag{A.4}$$

to write $R^{(1)}_{\mu\nu}$ in a different form.

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}(\Box h_{\mu\nu} + \bar{\nabla}_\mu \bar{\nabla}_\nu h + 2R^{\lambda}{}_{\mu}{}^{\alpha}{}_{\nu} h^{\lambda\alpha} - \bar{\nabla}_\mu \bar{\nabla}_\alpha h_\nu{}^\alpha - \bar{\nabla}_\nu \bar{\nabla}_\alpha h_\mu{}^\alpha - \bar{R}^\lambda{}_\mu h_{\nu\lambda} - \bar{R}^\lambda{}_\nu h_{\mu\lambda}). \tag{A.5}$$

The Einstein tensor can then be simplify calculated.

$$\begin{aligned} G^{(1)}_{\mu\nu} =& -\frac{1}{2}(\Box h_{\mu\nu} + \bar{\nabla}_\mu \bar{\nabla}_\nu h + 2R^{\lambda}{}_{\mu}{}^{\alpha}{}_{\nu} h^{\lambda\alpha} - \bar{\nabla}_\mu \bar{\nabla}_\alpha h_\nu{}^\alpha - \bar{\nabla}_\nu \bar{\nabla}_\alpha h_\mu{}^\alpha \\ & - \bar{R}^\lambda{}_\mu h_{\nu\lambda} - \bar{R}^\lambda{}_\nu h_{\mu\lambda} - \frac{1}{2}\bar{g}_{\mu\nu}(\bar{\nabla}^\alpha \bar{\nabla}^\beta h_{\alpha\beta} - \Box h - \bar{R}^{\alpha\beta} h_{\alpha\beta})) \end{aligned} \tag{A.6}$$

We shall now look at the matter part of Einstein's equation. We have:

$$\begin{aligned} T_{\mu\nu} &= F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} && \text{(A.7)} \\ &= g^{\alpha\rho}F_{\mu\rho}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}g^{\alpha\gamma}g^{\beta\delta}F_{\alpha\beta}F_{\gamma\delta} && \text{(A.8)} \\ &= (\bar{g}^{\alpha\rho} - h^{\alpha\rho})(\bar{F}_{\mu\rho} + f_{\mu\rho})(\bar{F}_{\nu\alpha} + f_{\nu\alpha}) && \text{(A.9)} \\ &- \frac{1}{4}(\bar{g}_{\mu\nu} + h_{\mu\nu})(\bar{g}^{\alpha\gamma} - h^{\alpha\gamma})(\bar{g}^{\beta\delta} - h^{\beta\delta})(\bar{F}_{\alpha\beta} + f_{\alpha\beta})(\bar{F}_{\gamma\delta} + f_{\gamma\delta})) \\ &= \bar{T}_{\mu\nu} + T^{(1,h)}_{\mu\nu} + T^{(1,f)}_{\mu\nu} && \text{(A.10)} \end{aligned}$$

where $\bar{T}_{\mu\nu}$ denotes the zeroth order (background) stress energy tensor and

$$T^{(1,h)}_{\mu\nu} = -\bar{F}_{\mu\rho}\bar{F}_{\nu\alpha}g^{\alpha\sigma}g^{\rho\gamma}h_{\sigma\gamma} - \frac{1}{4}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta}h_{\mu\nu} + \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\gamma}\bar{g}^{\beta\sigma}\bar{g}^{\delta\lambda}h_{\sigma\lambda}\bar{F}_{\alpha\beta}\bar{F}_{\gamma\delta} \tag{A.11}$$

$$= -[\bar{g}^{\alpha\sigma}\bar{g}^{\rho\gamma}(\bar{F}_{\mu\rho}\bar{F}_{\nu\alpha} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\beta\lambda}\bar{F}_{\alpha\beta}\bar{F}_{\rho\lambda})h_{\sigma\gamma} + \frac{1}{4}\bar{F}_{\alpha\beta}\bar{F}^{\alpha\beta}h_{\mu\nu}] \tag{A.12}$$

and

$$T^{(1,f)}_{\mu\nu} = \bar{F}_{\mu\rho}f_{\nu\alpha}\bar{g}^{\alpha\rho} + \bar{F}_{\nu\alpha}f_{\mu\rho}\bar{g}^{\alpha\rho} - \frac{1}{4}(\bar{g}_{\mu\nu}\bar{g}^{\gamma\alpha}\bar{g}^{\beta\delta}f_{\alpha\beta}\bar{F}_{\gamma\delta} + \bar{g}_{\mu\nu}\bar{g}^{\gamma\alpha}\bar{g}^{\beta\delta}f_{\gamma\delta}\bar{F}_{\alpha\beta}) \tag{A.13}$$

$$= \bar{g}^{\alpha\rho}(\bar{F}_{\mu\rho}f_{\nu\alpha} + \bar{F}_{\nu\rho}f_{\mu\alpha}) - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\gamma\alpha}\bar{g}^{\beta\delta}f_{\alpha\beta}\bar{F}_{\gamma\delta} \tag{A.14}$$

We shall now consider Maxwell's equation:

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial\mu}(\sqrt{-g}F^{\mu\nu}) = 0 \tag{A.15}$$

Using the identity that $\sqrt{-g} = \sqrt{-\bar{g}}(1 + \frac{1}{2}h)$, where $h = g^{\mu\nu}h_{\mu\nu}$, we can expand:

$$\frac{\partial}{\partial\mu}(\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}\bar{F}_{\alpha\beta}) \tag{A.16}$$

$$= \frac{\partial}{\partial\mu}(\sqrt{-\bar{g}}(1 + \frac{1}{2}h)(\bar{g}^{\mu\alpha} - h^{\mu\alpha})(\bar{g}^{\nu\beta} - h^{\nu\beta})(\bar{F}_{\alpha\beta} + f_{\alpha\beta})) \tag{A.17}$$

$$= \frac{\partial}{\partial\mu}(\sqrt{-\bar{g}}\bar{F}^{\mu\nu}) + \frac{\partial}{\partial\mu}(\frac{1}{2}h\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}\bar{F}_{\alpha\beta} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}f_{\alpha\beta} - \bar{g}^{\mu\alpha}h^{\nu\beta}\bar{F}_{\alpha\beta} - \bar{g}^{\nu\beta}h^{\mu\alpha}\bar{F}_{\alpha\beta}) \tag{A.18}$$

The first term equals zero to because the Maxwell equation is satisfied to zeroth order therefore

$$\frac{\partial}{\partial\mu}(\frac{1}{2}h\bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}F_{\alpha\beta} + \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta}f_{\alpha\beta} - \bar{g}^{\mu\alpha}h^{\nu\beta}F_{\alpha\beta} - \bar{g}^{\nu\beta}h^{\mu\alpha}F_{\alpha\beta}) = 0 \tag{A.19}$$

A.2 EFT equations of Motion

In this section I will calculate the additional contribution to the equations of motion coming from the $R_{\mu\nu}R^{\mu\nu}$ term in the action (2.35).

Starting with the well known Palatini Identity:

$$\delta R_{\mu\nu} = \nabla_\gamma \delta\Gamma^\gamma_{\mu\nu} - \nabla_\nu \delta\Gamma^\gamma_{\mu\gamma} \tag{A.20}$$

The variation of the Christoffel connection $\delta\Gamma$, unlike the Christoffel connection, is a tensor because the inhomogenous parts of the transformation cancel eachother. It is then convenient to work in a local inertial frame, in which

$$\begin{aligned}\delta\Gamma^\gamma_{\mu\nu} &= \frac{1}{2}\delta(g^{\gamma\alpha}(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu})) \\ &= \frac{1}{2}\delta g^{\gamma\alpha}(\partial_\nu g_{\alpha\mu} + \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}) + \frac{1}{2}g^{\gamma\alpha}(\partial_\nu \delta g_{\alpha\mu} + \partial_\mu \delta g_{\alpha\nu} - \partial_\alpha \delta g_{\mu\nu})\end{aligned}$$

Since we are working in a local inertial frame, all partial derivatives can now be replaced by covariant derivatives. The covariant derivative of the metric is zero therefore the first term disappears. Hence we are left we only the term:

$$\delta\Gamma^{\gamma}_{\mu\nu} = \frac{1}{2}g^{\gamma\alpha}(\nabla_\nu \delta g_{\alpha\mu} + \nabla_\mu \delta g_{\alpha\nu} - \nabla_\alpha \delta g_{\mu\nu}) \tag{A.21}$$

Re-inserting this into equation(A.20), we get:

$$\begin{aligned}\delta R_{\mu\nu} &= \frac{1}{2}\nabla_\gamma g^{\gamma\alpha}(\nabla_\nu \delta g_{\alpha\mu} + \nabla_\mu \delta g_{\alpha\nu} - \nabla_\alpha \delta g_{\mu\nu}) - \frac{1}{2}\nabla_\nu g^{\gamma\alpha}(\nabla_\gamma \delta g_{\alpha\mu} + \nabla_\mu \delta g_{\alpha\gamma} - \nabla_\alpha \delta g_{\mu\gamma}) \\ &= \frac{1}{2}g^{\gamma\alpha}(\nabla_\gamma\nabla_\nu \delta g_{\alpha\mu} + \nabla_\gamma\nabla_\mu \delta g_{\alpha\nu} - \nabla_\gamma\nabla_\alpha \delta g_{\mu\nu} - \nabla_\nu\nabla_\gamma \delta g_{\alpha\mu} - \nabla_\nu\nabla_\mu \delta g_{\alpha\gamma} + \nabla_\nu\nabla_\alpha \delta g_{\mu\gamma})\end{aligned}$$

The 4^{th} and 6^{th} terms cancel leaving

$$\delta R_{\mu\nu} = \frac{1}{2}g^{\gamma\alpha}(\nabla_\gamma\nabla_\nu \delta g_{\alpha\mu} + \nabla_\gamma\nabla_\mu \delta g_{\alpha\nu} - \nabla_\gamma\nabla_\alpha \delta g_{\mu\nu} - \nabla_\nu\nabla_\mu \delta g_{\alpha\gamma}) \tag{A.22}$$

We can now calculate the modification to the equations of motion coming from the $R_{\mu\nu}R^{\mu\nu}$ term.

$$\delta(\sqrt{-g}R_{\mu\nu}R^{\mu\nu}) = \sqrt{-g}(\delta R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu}\delta(g^{\mu\rho}g^{\sigma\nu}R_{\rho\sigma}) - \frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu}\delta g^{\mu\nu}) \tag{A.23}$$

$$= \sqrt{-g}(2R^{\mu\nu}\delta R_{\mu\nu} + 2R_\mu{}^\sigma R_{\nu\sigma}\delta g^{\mu\nu} - \frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu}\delta g^{\mu\nu}) \tag{A.24}$$

Substituting (A.22) into $2R^{\mu\nu}\delta R_{\mu\nu}$ we get:

$$2R^{\mu\nu}\delta R_{\mu\nu} = R^{\mu\nu}g^{\gamma\alpha}(\nabla_\gamma\nabla_\nu \delta g_{\alpha_\mu} + \nabla_\gamma\nabla_\mu \delta g_{\alpha\nu} - \nabla_\nu\nabla_\mu \delta g_{\gamma\alpha} - \nabla_\gamma\nabla_\alpha \delta g_{\mu\nu}) \tag{A.25}$$

$$= R^{\mu\nu}(2\nabla^\alpha\nabla_\nu \delta g_{\alpha\mu} - g^{\gamma\alpha}\nabla_\nu\nabla_\mu \delta g_{\gamma\alpha} - \Box \delta g_{\mu\nu}) \tag{A.26}$$

Remembering that we are doing this variation inside an integral, we can re-insert (A.27) into the integral and integrate by parts discarding the boundary terms as we have done previously.

$$\int d^4x\sqrt{-g} \quad 2g^{\alpha\beta}\nabla_\nu\nabla_\beta R^{\mu\nu}\delta g_{\alpha\mu} - g^{\gamma\alpha}\nabla_\mu\nabla_\nu R^{\mu\nu}\delta g_{\gamma\alpha} - \Box R^{\mu\nu}\delta g_{\mu\nu} \tag{A.27}$$

Using the identities $\delta g^{\mu\nu} g_{\mu\nu} + g^{\mu\nu}\delta g_{\mu\nu} = 0$ and $\delta g^{\mu\alpha} g_{\mu\nu} + g^{\mu\alpha}\delta g_{\mu\nu} = 0$, we can put in the form:

$$= \int d^4x\sqrt{-g} \quad -2g_{\alpha\mu}\nabla_\nu\nabla_\beta R^{\mu\nu}\delta g^{\alpha\beta} + g_{\gamma\alpha}\nabla_\mu\nabla_\nu R^{\mu\nu}\delta g^{\gamma\alpha} + \Box R_{\mu\nu}\delta g^{\mu\nu} \tag{A.28}$$

$$= \int d^4x\sqrt{-g} \quad -2\nabla_\alpha\nabla_\nu R_\mu{}^\alpha\delta g^{\mu\nu} + g_{\mu\nu}\nabla_\alpha\nabla_\beta R^{\alpha\beta}\delta g^{\mu\nu} + \Box R_{\mu\nu}\delta g^{\mu\nu} \tag{A.29}$$

Using the contracted Bianchi Identities

$$\nabla_\nu R^\nu{}_\mu = \frac{1}{2}\nabla_\mu R \tag{A.30}$$

We can simplify $\nabla_\alpha\nabla_\beta R^{\alpha\beta}$ as

$$\nabla_\alpha\nabla_\beta R^{\alpha\beta} = \frac{1}{2}\Box R \tag{A.31}$$

Therefore the final result for

$$\begin{aligned}&\delta\int d^4x\sqrt{-g}\, R_{\mu\nu}R^{\mu\nu}\\ &= \int d^4x\sqrt{-g}\,(2R_\mu{}^\sigma R_{\nu\sigma} - \frac{1}{2}R_{\alpha\beta}R^{\alpha\beta}g_{\mu\nu} - 2\nabla_\alpha\nabla_\nu R_\mu{}^\alpha + \frac{1}{2}g_{\mu\nu}\Box R + \Box R_{\mu\nu})\delta g^{\mu\nu}\end{aligned} \tag{A.32}$$

A.3 Modified Zerilli Equations

In this appendix, I will roughly give the steps used to put the electromagnetic and metric perturbations in the form (4.29, 4.30). Similarly to GR case, we get three equations from Einstein's equations and one from Maxwell's equations. The three Einsteins equations can be written compactly as:

$$a_1h_0 + a_2h_0^{(0,1)} + a_3h_0^{(0,2)} + a_4h_1^{(1,0)} + a_5h_1^{(1,1)} + \epsilon\left(\sum_{ij}(a_{0ij}h_0^{(i,j)} + a_{1ij}h_1^{(i,j)}\right) = a_6f_{12}, \tag{A.33}$$

$$b_1h_1 + b_2h_1^{(2,0)} + b_3h_0^{(1,0)} + b_4h_0^{(1,1)} + \epsilon\left(\sum_{ij}(b_{0ij}h_0^{(i,j)} + b_{1ij}h_1^{(i,j)}\right) = b_5f_{02}, \tag{A.34}$$

$$c_1 h_1 + c_2 h_1^{(0,1)} + c_3 h_0^{(1,0)} + \epsilon \left(\sum_{ij} (c_{0ij} h_0^{(i,j)} + c_{1ij} h_1^{(i,j)} \right) = 0, \tag{A.35}$$

where the coefficients, a'_{ij}, b'_{ij} and c'_{ij} are all functions of r. I have also denoted $\partial_t^{n_1} \partial_r^{n_2} h_i^{(n1.n2)}$ as $h_i^{(n1,n2)}$. Maxwell's equation is only modified by the change in background so it form is very similar to that of (4.21). I will focus on showing how to derive (4.29) - the basic procedure is essentially the same for (4.30). As in GR, the first equation is satisfied if both the second and the third are (by conservation of energy momentum tensor), so we only need to focus on the second and third equations. Again the third equation can be used to constrain h_0 in terms of h_1. This is not immediately obvious: the $O(\epsilon)$ part of this equation is rather complicated containing a mixture of the functions h_1 and h_0 and their derivatives. Fortunately it only actually contains h_0 with at least one time derivative i.e $c_{00j} = 0$. Remembering that we only working to $O(\epsilon)$ in the expansion, we can substitute the zeroth order form of $h_0^{(1,0)}$ into all higher derivative in order to get $h_0^{(1,0)}$ purely as a function of h_1 and its derivatives.

$$c_3 h_0^{(1,0} = -c_1 h_1 - c_2 h_1^{(0,1)} - \epsilon \sum_{ij} c'_{1ij} h_1^{(i,j)} \tag{A.36}$$

Similarly (A.35) only contains h_0 with at least one time derivative, i.e $b_{00j} = 0$. Therefore (A.36), can be substituted into (A.34) in order to get an equation just in terms of h_1 and its derivatives only. The $O(\epsilon)$ part of this equation will contain higher order derivatives of h_1 however these can be removed by substituting the lower order equations of motion. When this is all said and done we get an equation of the form:

$$d_1 h_1 + d_2 h_1^{(0,1)} + d_3 h_1^{(2,0)} + d_4 h_1^{(0,2)} + \epsilon (d_5 h_1 + d_6 h_1^{(0,1)} + d_7 h_2^{(2,0)} + d_8 f_{lm} + d_9 f_{lm}^{(2,0)}) = 0, \tag{A.37}$$

which we can then Fourier transform as in (4.22). The final task is then to subsitute the the zereoth order Maxwell equations into $f_{lm}^{(2,0)}$ to get our equation into the form shown

A.4 Explicit Expressions from Modified Zerilli Equation

$$f_{h1} = -c\frac{3r_Q^2 r_+^2}{r^4} \tag{A.38}$$

$$V_{GR} = J - \frac{3r_s}{r} + \frac{4r_Q^2}{r^2} \tag{A.39}$$

$$V_{HDO} = c\frac{r_+^2(r_Q^2 + r(-r_s + r))(44r_Q^2 + r_Q^2 r(-48r_s + (68 - 11J + 3J^2)r))}{r^8} \tag{A.40}$$

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